Symmetric Truncations of the Shallow Water Equations

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Abstract

Conservation of potential vorticity in Eulerian fluids reflects particle interchange symmetry in the Lagrangian fluid version of the same theory. The algebra associated with this symmetry in the shallow water equations is studied here, and we give a method for truncating the degrees of freedom of the theory which preserves a maximal number of invariants associated with this algebra. The symmetry associated with keeping N modes of the shallow water flow is SU(N). In the limit where the number of modes goes to infinity ($N \to \infty$), all the conservation laws connected with potential vorticity conservation are recovered. We also present a Hamiltonian which is invariant under this truncated symmetry and which reduces to the familiar shallow water Hamiltonian when $N \to \infty$. All this provides a finite dimensional framework for numerical work with the shallow water equations which preserves not only energy and enstrophy but all other known conserved quantities consistent with the finite number of degrees of freedom. The extension of these ideas to other nearly two dimensional flows is discussed.

1 Introduction

Many geophysical problems are naturally decomposed into a many layered approximation with each layer governed by the so-called shallow water equations [1]. These equations take the fluid density to be constant in each layer, and because the horizontal dimensions are assumed much larger than the vertical, hydrostatic balance is taken to hold in each layer separately. Vertical variations in each layer are ignored in the two dimensional horizontal velocity $\mathbf{v}(\mathbf{x}, t) = [v_1(x, y, t), v_2(x, y, t)]$ ($\mathbf{x} = (x, y)$), and incompressibility

$$\nabla \cdot \mathbf{v}(x, t) + \frac{\partial v_3(\mathbf{x}, z, t)}{\partial z} = 0, \tag{1}$$

determines the vertical velocity $v_3(\mathbf{x}, z, t)$. Using local hydrostatic balance, the pressure is eliminated in terms of the thickness of the vertical layer, $h(\mathbf{x}, t)$, and $h(\mathbf{x}, t)$ becomes the third dependent dynamical variable for the reduced system.

The evolution equations for $\mathbf{v}(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$ serve both as a useful model for the dynamics in a thin layer of fluid and as an important ingredient in more complicated models of the whole atmosphere or ocean [2]. In complex models which attempt to represent the full dynamics of the atmosphere, for example, one must add to the basic shallow water equations additional dynamics to describe radiative transfer, internal waves, cloud formation, relevant chemistry, etc. Whatever the goal of the dynamics of the shallow water equations, if one is to solve these equations some form of truncation of the infinite degrees of freedom must be made to progress numerically. Truncations directly in Eulerian or Lagrangian configuration space or in the dual Fourier space fail to preserve all the conservation laws respected by the underlying particle interchange symmetry of the Lagrangian theory which exhibits itself in the conservation of potential vorticity in Eulerian fluid mechanics. These latter remarks, of course, apply only when the shallow water flow is inviscid, as we shall

assume throughout this paper. We shall have a few remarks to make at the end of this paper about the potential use of our results for the case with friction.

In this paper we take up the much studied subject of shallow water equations with the goal of providing a truncation of the degrees of freedom from infinity to a finite number using a method which preserves the maximum number of conserved quantities consistent with the reduction in the number of degrees of freedom. When this number returns to infinity, that is when the truncation is removed, the theory preserves all the quantities associated with potential vorticity conservation. Our work takes place in the Lagrangian formulation of the theory. The truncation is made in the Fourier space of variables dual to the Lagrangian labels of fluid particles. The algebra associated with the symmetry of particle interchange is altered as part of the truncation, and the finite number of Casimir invariants of the new algebra, which is SU(N) and thus familiar, replace the infinite number of conserved quantities following from potential vorticity conservation. In the limit $N \rightarrow \infty$, the usual conserved quantities are recovered.

The methods we shall use we first learned from two sources. One is the work by Fairlie, *et al* [3] on finite algebras in string theory, and the other is an application of those methods to the two dimensional Euler equations independently invented by Rouhi [4] and Zeitlin [5]. The latter application may be quite interesting in other geophysical applications where two dimensional Eulerian flows are studied, but we have not pursued that line of investigation. We have analyzed the shallow water equations, as presented here, both for their interest as indicated, and also because they have numerous useful formal similarities with internal wave dynamics and with surface wave physics. We will indicate these similarities at the end of this paper. Our work here is also in planar geometries. The extension to flows on a sphere, while algebraically complicated, is more or less straightforward in concept as seen

in the paper of Hoppe [6].

The shallow water equations and their numerical solution using various truncations has become a subject of renewed interest of late because of the work to place these equations and their more complex forms on parallel processing machines [7]. The goal of that effort is to build numerically efficient climate models for investigations of very long times (thousands of simulated years) and/or issues requiring very high spatial resolution. We expect that the truncation presented here, which by its formulation preserves as much of the original symmetry as possible of the structure of the shallow water equations, will prove an attractive alternative to straightforward finite element, discrete spatial grid, or spectral methods for these equations.

In the next Section we review the shallow water equations in Eulerian and Lagrangian formulation and write down the algebraic structure associated with particle interchange symmetry. The third Section is devoted to the $SU(N)$ truncation of the theory in Lagrangian formulation and also presents the truncated Hamiltonian for the shallow water flow. The final Section has our comments about further uses of our observations in other problems of geophysical interest and contains the summary of our present work.

2 Shallow Water Theory

2.1 Equations of Motion and Symmetry

The Eulerian shallow water equations govern the evolution of a two dimensional horizontal velocity $\mathbf{v}(\mathbf{x}, t)$ in a fluid of vertical thickness $h(\mathbf{x}, t)$ of the fluid via the familiar evolution equations

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + \mathbf{v}(x, t) \cdot \nabla \, \mathbf{v}(x, t) \;\; = \;\; -g \nabla h(\mathbf{x}, t)$$

$$\frac{\partial h(\mathbf{x},t)}{\partial t} + \nabla \cdot [h(\mathbf{x},y)\mathbf{v}(\mathbf{x},t)] \;=\; 0, \tag{2}$$

where g is the gravitational constant and $\nabla = [\partial_1, \partial_2]$. If the frame is rotating about the z-axis at angular velocity $f/2$, a term $\mathbf{v}(\mathbf{x},t) \times \hat{z} f(\mathbf{x})$ appears in the equation for $\mathbf{v}(x,t)$. As indicated above these equations follow from the three dimensional Euler equations of a thin, homogeneous fluid with hydrostatic balance in each layer determining the pressure $p(\mathbf{x},z,t)$ in that layer in terms of the thickness $p(\mathbf{x},z,t) = g[h(\mathbf{x},t) - z]$.

The total energy

$$H_E(\mathbf{v},h) = \frac{1}{2}\int d^2x\,[|\mathbf{v}(\mathbf{x},t)|^2 + h(\mathbf{x},t)^2], \tag{3}$$

is conserved by solutions to these equations and the Eulerian potential vorticity

$$q_E(\mathbf{x},t) = \frac{\hat{z} \cdot \nabla \times \mathbf{v}(x,t)}{h(\mathbf{x},t)} \tag{4}$$

satisfies

$$\frac{\partial q_E(\mathbf{x},t)}{\partial t} + \mathbf{v}(x,t) \cdot q_E(\mathbf{x},t) = 0, \tag{5}$$

so

$$\int d^2x\, h(\mathbf{x},t) G(q_E(\mathbf{x},t)) \tag{6}$$

is time independent for arbitrary $G(q_E)$.

These conservation laws arise from the particle interchange symmetry exhibited by the canonical or Lagrangian formulation of the theory. In Lagrangian formulation, the dynamical variables are the particle position $\mathbf{Y}(\mathbf{r},t)$ at every particle label r and time and the conjugate momentum $\mathbf{P}(\mathbf{r},t)$. In terms of these variables the Hamiltonian reads

$$H_L(\mathbf{Y},\mathbf{P}) = \frac{1}{2}\int d^2r\,[|\mathbf{P}(\mathbf{r},t)|^2 + gJ(\mathbf{Y}(\mathbf{r},t))^{-1}], \tag{7}$$

where the Jacobian

$$J(\mathbf{Y}(\mathbf{r},t)) = \frac{\partial(\mathbf{Y})}{\partial(\mathbf{r})} \tag{8}$$

has been introduced. In these equations and below a Lagrangian density $\rho_0(\mathbf{r})$ which describes the distribution of mass among the particles labeled by $\mathbf{r}$ has been absorbed into the definition of the labels themselves [8].

The evolution in time of any functional $A[\mathbf{Y}(\mathbf{r},t),\mathbf{P}(\mathbf{r},t)]$ follows from Hamiltonian's equations

$$\frac{\partial A[\mathbf{Y}(\mathbf{r},t),\mathbf{P}(\mathbf{r},t)]}{\partial t} = \{A[\mathbf{Y}(\mathbf{r},t),\mathbf{P}(\mathbf{r},t)], H(\mathbf{Y},\mathbf{P})\}, \tag{9}$$

introducing the canonical Poisson brackets between functionals $A(\mathbf{Y},\mathbf{P})$ and $B(\mathbf{Y},\mathbf{P})$

$$\{A(\mathbf{Y},\mathbf{P}), B(\mathbf{Y},\mathbf{P})\} = \int d^2r\, [\frac{\partial\delta A(\mathbf{Y},\mathbf{P})}{\partial\delta\mathbf{Y}(\mathbf{r},t)}\frac{\partial\delta \mathbf{B}(\mathbf{Y},\mathbf{P})}{\partial\delta\mathbf{P}(\mathbf{r},t)} - (A \leftrightarrow B)]. \tag{10}$$

The fundamental canonical brackets between $\mathbf{Y}(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$

$$\{Y_a(\mathbf{r},t), P_b(\mathbf{r}',t)\} = \delta_{ab}\delta^2(\mathbf{r}-\mathbf{r}') \tag{11}$$

follow from this bracket, and we shall use it extensively below.

In this notation the potential vorticity takes the form

$$q(\mathbf{r},t) = \epsilon_{ab}\frac{\partial Y_\alpha(\mathbf{r},t)}{\partial r_a}\frac{\partial P_\alpha(\mathbf{r},t)}{\partial r_b}, \tag{12}$$

where the indices $a, b, \alpha, \dots$ run over 1 and 2, and repeated indices are summed over. ϵ_{ab} is the completely antisymmetric symbol in two dimensions. The conservation law is

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = 0, \tag{13}$$

and all derivatives in Lagrangian formulation are taken at fixed $\mathbf{r}$. The integrals

$$C[\mathbf{Y},\mathbf{P}] = \int d^2r\, G(q(\mathbf{r},t)) \tag{14}$$

are clearly constant in time for any $G(q)$.

The generators of the particle interchange symmetry are determined by examining the invariance of the action defining Hamilton's principle

$$\text{ACTION} = \int_{t_1}^{t_2} \int d^2r \, \frac{1}{2}[|\frac{\partial \mathbf{Y}(\mathbf{r},t)}{\partial t}|^2 - gJ^{-1}], \tag{15}$$

under variations in $\mathbf{r}$ at fixed $\mathbf{Y}(\mathbf{r},t)$ and fixed J. These variations must be of the form [9]

$$\delta r_a = \epsilon_{ab} \frac{\partial A(\mathbf{r})}{\partial r_b} \tag{16}$$

with $A(\mathbf{r})$ an arbitrary function of $\mathbf{r}$. The time independent functional $C[\mathbf{Y},\mathbf{P}]$ defined above has the property that

$$\{r_a, C[\mathbf{Y},\mathbf{P}]\} = \epsilon_{ab} \frac{\partial}{\partial r_b} \frac{\partial G(q)}{\partial q}, \tag{17}$$

which is of the correct form. So the global generators of the particle interchange symmetry are just integrals of arbitrary functions of the potential vorticity. The algebra associated with the particle interchange symmetry is rather rich. The potential vorticity does not Poisson commute with itself, but the Poisson brackets of the $q(\mathbf{r},t)$ do close on themselves. We turn to this algebra now.

2.2 Potential Vorticity Algebra

To study the algebra of the potential vorticity we have found it most straightforward to work in the Fourier space dual to the particle labels $\mathbf{r}$. We continue to work in planar geometry and limit horizontal space to a box of size $L \times L$. In this box we introduce Fourier transforms as follows: for functions $f(\mathbf{r})$ we write

$$f(\mathbf{r}) = \sum_{\mathbf{n}=-\infty}^{\infty} g(\mathbf{n}) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \tag{18}$$

and the inverse

$$g(\mathbf{n}) = \frac{1}{L^2} \int d^2r\, f(\mathbf{r}) \exp[-i\kappa \mathbf{n} \cdot \mathbf{r}], \tag{19}$$

where the vectors $\mathbf{n}, \mathbf{m}, \ldots$ are composed of integers $\mathbf{n} = [n_1, n_2]; n_a = 0, \pm 1, \pm 2, \ldots$, and $\kappa = 2\pi/L$. We take the Fourier decomposition of the canonical variables to be

$$\begin{aligned} \mathbf{Y}(\mathbf{r}) &= \frac{1}{L} \sum_{\mathbf{n}} \mathbf{Q}(\mathbf{n}) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}] \\ \mathbf{P}(\mathbf{r}, t) &= \frac{1}{L} \sum_{\mathbf{n}} \mathit{\Pi}(\mathbf{n}) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \end{aligned} \tag{20}$$

which gives the Poisson brackets in Fourier space

$$\{Q_a(\mathbf{n}), \Pi_b(\mathbf{m})\} = \delta_{ab} \delta_{\mathbf{n},-\mathbf{m}}. \tag{21}$$

Using this Fourier transformed set of canonical variables we have for the potential vorticity

$$q(\mathbf{r}) = \frac{(2\pi)^2}{L^4} \sum_{\mathbf{n}} \varsigma(\mathbf{n}) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \tag{22}$$

with

$$\varsigma(\mathbf{n}) = \sum_{\mathbf{m}} \mathbf{n} \times \mathbf{m}\, \Pi_a(\mathbf{m}) Q_a(\mathbf{n} - \mathbf{m}). \tag{23}$$

We have defined the quantity $\mathbf{n} \times \mathbf{m} = n_1 m_2 - n_2 m_1$ is this equation, and the normalization for $\varsigma(\mathbf{n})$ has been chosen to make the last formula and many to follow as simple as possible.

The algebraic properties of $\varsigma(\mathbf{n})$ are seen in the Poisson brackets

$$\begin{aligned} \{\varsigma(\mathbf{n}), Q_a(\mathbf{m})\} &= \mathbf{n} \times \mathbf{m}\, Q_a(\mathbf{n} + \mathbf{m}), \\ \{\varsigma(\mathbf{n}), \Pi_a(\mathbf{m})\} &= \mathbf{n} \times \mathbf{m}\, \Pi_a(\mathbf{n} + \mathbf{m}), \end{aligned} \tag{24}$$

and

$$\{\varsigma(\mathbf{n}), \varsigma(\mathbf{m})\} = \mathbf{n} \times \mathbf{m}\, \varsigma(\mathbf{n} + \mathbf{m}). \tag{25}$$

The last Poisson bracket exhibits the structure of the particle interchange algebra and shows it to be an infinite dimensional algebra, as we might expect since it comes from a continuum set of operations on label space $\mathbf{r}$. This set of continuum operations translates via the Fourier transform into a discrete infinity of operation in $\mathbf{n}$ space.

We will also require the properties of the Jacobian under Poisson commutation with the potential vorticity $\varsigma(\mathbf{n})$ since we will be interested in construction a Hamiltonian invariant under the particle interchange symmetry. The Hamiltonian of the shallow water equations is invariant under this symmetry, which is our starting point for all this after all, and this invariance is the same statement as the time independence of $q(\mathbf{r}, t)$

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = \{q(\mathbf{r},t), H(\mathbf{Y},\mathbf{P})\} = 0. \tag{26}$$

When we truncate the Fourier transform variables in just a moment we will want to retain this invariance under the truncated potential vorticity. The transformation properties of the Jacobian under $\varsigma(\mathbf{n})$ will be of interest in this.

The Fourier decomposition of the Jacobian is

$$\begin{aligned} J(\mathbf{Y}(\mathbf{r},t)) &= \frac{\partial(\mathbf{Y})}{\partial(\mathbf{r})} \\ &= \frac{(2\pi)^2}{L^4} \sum_{\mathbf{n}} \rho(\mathbf{n}) \exp[i\kappa\mathbf{n}\cdot\mathbf{r}], \end{aligned} \tag{27}$$

which gives

$$\begin{aligned} \rho(\mathbf{n}) &= \frac{1}{2} \sum_{\mathbf{m},\mathbf{m}'} \mathbf{m}' \times \mathbf{m}\, \mathbf{Q}(m) \times \mathbf{Q}(\mathbf{m}') \delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'} \\ &= \sum_{\mathbf{m}} \mathbf{n} \times \mathbf{m}\, Q_1(\mathbf{m}) Q_2(\mathbf{n}-\mathbf{m}). \end{aligned} \tag{28}$$

The Poisson bracket of $\varsigma(n)$ and $\rho(\mathbf{m})$ is then

$$\{\varsigma(\mathbf{n}), \rho(\mathbf{m})\} = \mathbf{n} \times \mathbf{m}\, \rho(\mathbf{m}+\mathbf{n}). \tag{29}$$

One may summarize these brackets as saying that under the infinite algebra of the potential vorticity $\varsigma(\mathrm{n})$ the quantities $\mathrm{Q}(\mathrm{m}), \Pi(\mathrm{m}), \rho(\mathrm{m})$ and $\varsigma(m)$ itself transform as "vectors".

3 Truncation of the Modes

3.1 Algebra of the Truncated Modes: SU(N)

Our truncation of the particle interchange algebra is motivated by the idea of limiting the number of Fourier modes, but a simple cutoff on the components of the vectors n does not respect the Poisson brackets. Suppose we limit each component of our integer vectors $\mathrm{n} = [n_1, n_2]$ to $-M \leq n_a \leq M$, then the presence of the sum vectors $\mathrm{n} + \mathrm{m}$ in the Poisson brackets of $\varsigma(\mathrm{n})$ means that vectors in the range are mapped out of the range. To address this we note that restricting vectors in label space to a box of size $L \times L$ and making the Fourier transform, we have in effect mapped our space onto a torus by implicitly identifying the sides under the assumption of periodicity. If we were to formalize this periodicity by requiring all sums of integer vectors to lie within the range $[-M, M]$ by a modulo or remainder operation, we would still need to deal with the terms $\mathrm{n} \times \mathrm{m}$ that appear in all of the Poisson brackets with $\varsigma(\mathrm{n})$. These are the so called structure constants associated with the group properties of particle relabeling invariance, so it is suggestive that modifying them as well would be required to make a consistent theory of truncated Fourier modes. In particular one must address the Jacobi identity which provides the statement that the operations in question do close to form a group.

The set of operations which provides a consistent truncation of the modes comes from

changing the definition of the Fourier components of $q(\mathbf{r},t)$ by

$$\varsigma_N(\mathbf{n}) = \frac{1}{\kappa_N} \sum_{\mathbf{m}=-M}^{M} \sin[\kappa_N \mathbf{n} \times \mathbf{m}] \Pi_a(\mathbf{m}) Q_a(\mathbf{n}-\mathbf{m}), \tag{30}$$

where $N = 2M + 1$, all components $n_a, m_a, \ldots$ are restricted to [-M,M], and $\kappa_N = 2\pi/N$. In the limit $M \to \infty$ or equivalently $N \to \infty$, namely as $\kappa_N \to 0$, this definition of the potential vorticity is the same as in the original Fourier transform. The definition of the Fourier density for the Jacobian is modified to

$$\begin{aligned} \rho_N(\mathbf{n}) &= \frac{1}{2} \sum_{\mathbf{m},\mathbf{m}'=-M}^{M} \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{m}' \times \mathbf{m}]\, \mathbf{Q}(m) \times \mathbf{Q}(\mathbf{m}') \delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'} \\ &= \sum_{\mathbf{m}=-M}^{M} \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{n} \times \mathbf{m}]\, Q_1(\mathbf{m}) Q_2(\mathbf{n}-\mathbf{m}). \end{aligned} \tag{31}$$

The definitions of the $\mathbf{Q}(\mathbf{n})$ and $\mathbf{P}(\mathbf{n})$ are unchanged and the Poisson brackets between them are still

$$\{Y_a(\mathbf{n}), \Pi_b(\mathbf{m})\} = \delta_{ab} \delta_{0,\mathbf{m}+\mathbf{n}}, \tag{32}$$

with the rule that vector components out of [-M,M] are mapped back into the range.

Now the Poisson brackets among the $\varsigma_N(\mathbf{n})$ and the other variables $\mathbf{Y}(\mathbf{n}), \mathit{\Pi}(\mathbf{n}), \rho_N(\mathbf{n})$ are found to be

$$\begin{aligned} \{\varsigma_N(\mathbf{n}), Q_a(\mathbf{m})\} &= -\frac{1}{\kappa_N} \sum_{\mathbf{m}'=-M}^{M} \sin[\kappa_N \mathbf{n} \times \mathbf{m}'] \delta_{0,\mathbf{m}'+\mathbf{m}} Q_a(\mathbf{n}-\mathbf{m}') \\ &= \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{n} \times \mathbf{m}] Q_a(\mathbf{n}+\mathbf{m}), \end{aligned} \tag{33}$$

and

$$\{\varsigma_N(\mathbf{n}), \Pi_a(\mathbf{m})\} = \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{n} \times \mathbf{m}] \Pi_a(\mathbf{n}+\mathbf{m}), \tag{34}$$

$$\{\varsigma_N(\mathbf{n}), \rho_N(\mathbf{m})\} = \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{n} \times \mathbf{m}] \rho_N(\mathbf{n}+\mathbf{m}), \tag{35}$$

$$\{\varsigma_N(\mathbf{n}), \varsigma_N(\mathbf{m})\} = \frac{1}{\kappa_N} \sin[\kappa_N \mathbf{n} \times \mathbf{m}] \varsigma_N(\mathbf{n}+\mathbf{m}). \tag{36}$$

In each of these Poisson bracket relations we have used the trigonometric identity

$$\begin{aligned}\sin(a[\mathrm{m}\times\mathrm{m}'-\mathrm{m}\times\mathrm{n}])\sin(a[\mathrm{n}\times\mathrm{m}']) &+ \\ \sin(a[\mathrm{n}\times\mathrm{m}-\mathrm{n}\times\mathrm{m}']) &= \sin(a[\mathrm{n}\times\mathrm{m}])\sin(a[(\mathrm{n}+\mathrm{m})\times\mathrm{m}']),\end{aligned} \quad (37)$$

for an arbitrary (complex) constant a. For us $a=\kappa_N$.

This set of Poisson brackets defines a algebra which is SU(N) with the $\varsigma(n)$ as generators of infinitesimal SU(N) transformations. The truncation of the Fourier modes with the modification of the Poisson brackets now provides a consistent reduction from an infinite number of modes to N. The critical issue in checking this consistency is verifying that the Jacobi identity among Poisson brackets is satisfied, and with the change of structure constants from $\mathrm{n}\times\mathrm{m}\rightarrow\frac{1}{\kappa_N}\sin[\kappa_N\mathrm{n}\times\mathrm{m}]$ this is readily established.

We want to construct a Hamiltonian, H_N, now which is invariant under this SU(N) and becomes the shallow water Hamiltonian in the limit $N\rightarrow\infty$. For this purpose we need to investigate the Casimir invariants of the SU(N) algebra. These we generalize from the observations of Fairlie *et. al.* [10]. If the Poisson brackets of $\varsigma_N(\mathrm{n})$ with any function in Fourier space $f(\mathrm{m})$ is

$$\{\varsigma_N(\mathrm{n}),f(\mathrm{m})\}=\frac{1}{\kappa_N}\sin[\kappa_N\mathrm{n}\times\mathrm{m}]f(\mathrm{n}+\mathrm{m}), \quad (38)$$

then the following sums Poisson commute with $\varsigma_N(n)$

$$\begin{aligned}C_2(f) &= \sum_{\mathrm{n},\mathrm{m}=-M}^{M} f(\mathrm{n})f(\mathrm{m})\delta_{0,\mathrm{n}+\mathrm{m}}, \\ C_3(f) &= \sum_{\mathrm{n}_i=-M}^{M} f(\mathrm{n}_1)f(\mathrm{n}_2)f(\mathrm{n}_3)\exp[i\kappa_N(\mathrm{n}_1\times\mathrm{n}_2+\mathrm{n}_1\times\mathrm{n}_3+\mathrm{n}_2\times\mathrm{n}_3]\delta_{0,\mathrm{n}_1+\mathrm{n}_2\mathrm{n}_3}\end{aligned} \quad (39)$$

and similarly

$$C_{L+1}(f)=\sum_{\mathrm{n}_1,\ldots,\mathrm{n}_L=\ M}^{M}\prod_{\alpha<\beta}\exp[i\kappa_N\mathrm{n}_\alpha\times\mathrm{n}_\beta]f(\mathrm{n}_1)f(\mathrm{n}_2)\ldots f(\mathrm{n}_L)f(-(\mathrm{n}_1+\mathrm{n}_2+\ldots+\mathrm{n}_L)) \quad (40)$$

These sums go over in the limit $\kappa_N \to 0$ to the integrals

$$\int d^2r\, [f(\mathbf{r})]^{j+1}, \tag{41}$$

for $j = 1, \ldots L$. We will use this in constructing H_N.

3.2 The Truncated Hamiltonian: H_N

We begin by remembering how the full Lagrangian variable Hamiltonian

$$H(\mathrm{Q}(\mathrm{n}), \Pi(\mathrm{n})) = \frac{1}{2} \sum_{\mathrm{n}=-\infty}^{\infty} [\Pi_a(\mathrm{n})\Pi_a(-\mathrm{n})] + g \int d^2r\, J^{-1}, \tag{42}$$

conserves $\varsigma(\mathrm{n})$. The Poisson brackets of $\varsigma(\mathrm{n})$ with the first term is

$$\frac{1}{2} \sum_{\mathrm{m}=-\infty}^{\infty} [\mathrm{n} \times \mathrm{m}\, \Pi_a(\mathrm{m}+\mathrm{n})\Pi_a(-\mathrm{m}) - \mathrm{n} \times \mathrm{m}\, \Pi_a(\mathrm{m})\Pi_a(\mathrm{n}-\mathrm{m})], \tag{43}$$

which vanishes by symmetry; let $\mathrm{m} \to -\mathrm{m}$ in the second sum. The Poisson bracket of the second term is

$$\{\varsigma_N(\mathrm{n}), \int d^2r\, J^{-1}\} = \frac{1}{\kappa^2} \int d^2r \frac{\partial(\exp[-i\kappa\mathrm{n}\cdot\mathrm{r}], J^{-1})}{\partial(r_1, r_2)}, \tag{44}$$

which vanishes upon integration by parts.

When we change the structure constants in the Poisson brackets and truncate the modes so all n are in [-M,M], the Poisson bracket of $\varsigma_N(\mathrm{n})$ with the kinetic energy becomes

$$\begin{aligned}\{\varsigma_N(\mathrm{n}), \frac{1}{2} \sum_{\mathrm{m}=-M}^{M} \Pi_a(\mathrm{m})\Pi_a(-\mathrm{m})\} &= \frac{1}{\kappa_N} \sum_{\mathrm{m}=-M}^{M} \\ &\quad [\sin[\kappa_N \mathrm{n} \times \mathrm{m}] \Pi(\mathrm{n}+\mathrm{m}) \cdot \Pi(-\mathrm{m}) \\ &- \sin[\kappa_N \mathrm{n} \times \mathrm{m}] \Pi(\mathrm{m}) \cdot \Pi(\mathrm{n}-\mathrm{m})],\end{aligned} \tag{45}$$

which again vanishes because of symmetry.

The Jacobian term is more tricky. The Poisson bracket of $\varsigma_N(\mathrm{n})$ with

$$J_N(\mathrm{m}) = \frac{(2\pi)^2}{L^4} \sum_{\mathrm{n}=-M}^{M} \rho_N(\mathrm{m}) \exp[i\kappa_N \mathrm{n}\cdot\mathrm{m}], \tag{46}$$

is

$$\{\varsigma(\mathrm{n}), J_N(\mathrm{m})\} = \frac{N}{4\pi i} \exp[-i\kappa_N \mathrm{n}\cdot\mathrm{m}](J_N(\mathrm{m}+\hat{z}\times\mathrm{n}) - J_N(\mathrm{m}-\hat{z}\times\mathrm{n})). \tag{47}$$

Though in the limit as $\kappa_N \to 0$, this becomes the same as the continuum theory, it shows that a Hamiltonian constructed out of integrals over d^2r of $J^{\pm1}$ will not Poisson commute with $\varsigma(\mathrm{n})$.

We choose instead to represent the quantity J^{-1} which appears under the integral in the Hamiltonian as an expansion in powers around $J = J_0$. The full expansion is

$$\begin{aligned} \frac{1}{J} &= \frac{1}{J_0 - (J_0 - J)} \\ &= \frac{1}{J_0}\sum_{k=0}^{\infty}(1 - \frac{J}{J_0})^k, \end{aligned} \tag{48}$$

which we truncate at $k = N - 1$ to

$$\begin{aligned} \frac{1}{J} &\approx \frac{1}{J_0}\sum_{k=0}^{N-1}(1 - \frac{J}{J_0})^k \\ &= \frac{1}{J}(1 - (1 - \frac{J}{J_0})^N). \end{aligned} \tag{49}$$

Clearly for $(1 - \frac{J}{J_0})^N \ll 1$ this expression is essentially $\frac{1}{J}$. It is also a finite sum of powers of J, and this leads to the approximation we will use in H_N:

$$\begin{aligned} \frac{1}{J} &\approx \frac{1}{J}(1 - (1 - \frac{J}{J_0})^N) \\ &= \frac{1}{J_0}\sum_{k=1}^{N}(-\frac{J}{J_0})^{k-1} C_k^N, \end{aligned} \tag{50}$$

which should be good for $|1 - \frac{J}{J_0}| < 1$ or $0 < \frac{J}{J_0} < 2$, which will suit our purposes quite well.

The idea is then to approximate $\int d^2r \frac{1}{J}$ by

$$\frac{1}{J_0}\sum_{k=1}^{N} C_k^N \int d^2r(-\frac{J_N}{J_0})^{k-1}, \tag{51}$$

and then to replace $\int d^2r\, J_N^{p+1}$ by $\frac{(2\pi)^{2p+2}}{L^{2(2p+1)}}$ times

$$\begin{aligned} C_{p+1}(\rho_N) \quad = \quad & \sum_{\mathrm{n}_1,\ldots,\mathrm{n}_p=-M}^{M} \prod_{\alpha<\beta} \\ & \exp[i\kappa_N \mathrm{n}_\alpha \times \mathrm{n}_\beta]\rho_N(\mathrm{n}_1)\rho_N(\mathrm{n}_2)\ldots\rho_N(\mathrm{n}_p)\rho_N(-(\mathrm{n}_1+\mathrm{n}_2+\ldots+\mathrm{n}_p)), \end{aligned} \tag{52}$$

recalling that

$$\rho_N(\mathrm{m}) = \frac{L^4}{(2\pi N)^2}\sum_{\mathrm{n}=-M}^{M} J_N(\mathrm{n})\exp[-i\kappa_N \mathrm{m}\cdot\mathrm{n}]. \tag{53}$$

The Hamiltonian invariant under the action of the local SU(N) generators $\varsigma(\mathrm{n}))$ is then

$$H_N = \frac{1}{2}\sum_{\mathrm{n}=-M}^{M} \Pi(\mathrm{n})\cdot\Pi(-\mathrm{n}) + \frac{g}{2J_0}\sum_{k=1}^{N} C_k^N \frac{(2\pi)^{2k-2}}{L^{2(2k-3)}} C_{k-1}(\rho_N). \tag{54}$$

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